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A THEORY OF ALGEBRAIC INTEGRATION

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In this paper we study the problem of quantizing theories defined over a nonclassical configuration space. If one follows the path-integral approach, the first problem one is faced with is the one of definition of the integral over such spaces. We consider this problem and we show how to define an integration which respects the physical principle of composition of the probability amplitudes for a very large class of algebras.

1 Introduction and motivations

In the ordinary description of quantum mechanics the classical phase space gets promoted to a noncommutative algebra. However, in the path-integral quantization one still retains a classical phase space (but with a different physical interpretation). The situation changes in supersymmetric theories where one deals with a phase space including Grassmann variables [1]. Correspondingly one faces the problem of defining the integration over a nonclassical space. This of course is solved by means of the integration rules for Grassmann algebras introduced by Berezin [2]. These rules were originally introduced in relation to the quantization of Fermi fields and they were justified by the fact that they reproduce the perturbative expansion. In a more general setting one would like to know if there exists a principle underlying these rules. A possible answer is the translational invariance of the Berezin's rules. In fact the induced functional measure is again translational invariant and this has as a consequence the validity of the Schwinger quantum action principle. However, if one looks for generalizations of ordinary quantum mechanics, as for instance quantum groups, M-theory, etc, it is not clear whether one would insist on the validity of such a principle. We would feel more confident if we could rely upon something more fundamental. Our proposal is to use as a fundamental property the *combination law for probability amplitudes.* In the usual path integral approach this is a trivial consequence of the factorization properties of the functional measure, which in turn is equivalent to the completeness of the position eigenstates. In the general situation we are considering, the space of the eigenvalues

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of the position operator goes into a non classical space. By following the spirit of noncommutative geometry [3] we will deal with this situation by considering the space of the functions on the space, rather than the space itself (which in general has no particular meaning). Therefore, one has to know how to lift up the completeness relation from the space to the space of the functions. We can see how things work in the classical case. Consider an orthonormal set of states $|\psi_n\rangle$, then we can convert the completeness relation in the configuration space into the orthogonality relation for the functions $\psi_n(x) = \langle x|\psi\rangle$

$$\int \langle\psi_m|x\rangle\langle x|\psi_n\rangle = \int \psi_m^*(x)\psi_n(x)dx = \delta_{nm} \tag{1}$$

Notice that the completeness in the x-space can be reconstructed from this orthonormality relation and from the completeness of the states. The properties of the set $\{\psi_n(x)\}$ following from the completeness are

- The set $\{\psi_n(x)\}$ spans a vector space.
- The product $\psi_m(x)\psi_n(x)$ can be expressed as a linear combination of elements of the set $\{\psi_n(x)\}$.

In other words, the set $\{\psi_n(x)\}$ has the structure of an algebra. By using again the completeness we can write

$$\psi_n^*(x) = \sum_m C_{nm}\psi_m(x) \tag{2}$$

from which

$$\int \sum_m C_{nm}\psi_m(x)\psi_p(x)dx = \delta_{np} \tag{3}$$

In the following we will use this formula to define the integration over an algebra with basis elements $\{x_i\}$

$$\int_{(x)} \sum_j C_{ij}x_jx_k = \delta_{ik} \tag{4}$$

with C a matrix to be specified.

2 Algebras

An algebra is a vector space $\mathcal{A}$ equipped with a bilinear mapping $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$. We will work in a fixed basis $\{x_i\}$, with $i = 0, \cdots, n$ (we will consider also

the possibility of $n \to \infty$, or of a continuous index). The algebra product is determined by the structure constants f_{ijk}

$$x_i x_j = f_{ijk} x_k \tag{5}$$

An important tool is the algebra of the left and right multiplications. The associated matrices are defined by

$$R_a|x\rangle = |x\rangle a, \qquad \langle x|L_a = a\langle x|, \qquad a \in \mathcal{A} \tag{6}$$

with $\langle x| = (x_0, x_1, \cdots, x_n)$. It follows ($L_i \equiv L_{x_i}$)

$$(R_i)_{jk} = f_{jik}, \qquad (L_i)_{jk} = f_{ikj} \tag{7}$$

These matrices encode all the properties of the algebra. For instance, in the case of an associative algebra, the associativity condition can be expressed in the following three equivalent ways

$$L_i L_j = f_{ijk} L_k, \qquad R_i R_j = f_{ijk} R_k, \qquad [R_i, L_j^T] = 0 \tag{8}$$

In particular we see that L_i and R_i are representations of the algebra (regular representations).

3 Integration Rules

In the following, $\mathcal{A}$ will stay for an associative algebra with identity. We will say that $\mathcal{A}$ is a *self-conjugated* algebra, if the two regular representations corresponding to the right and left multiplications are equivalent, *i.e.*

$$R_i = C^{-1} L_i C \tag{9}$$

In this case one has

$$L_i|Cx\rangle = |Cx\rangle x_i, \qquad |Cx\rangle = C|x\rangle \tag{10}$$

We then define the integration over $\mathcal{A}$ by the requirement [4]

$$\int_{(x)} |Cx\rangle\langle x| = 1 \tag{11}$$

The problem here is that by using the algebra product, we get

$$\int_{(x)} C_{ij} x_j x_k = C_{ij} f_{jkl} \int_{(x)} x_l = \delta_{ik} \tag{12}$$

but these are $(n+1)^2$ equations for the $n+1$ unknown quantities $\int_{(x)} x_l$. However, by assuming $x_0 = I$, and by taking $k=0$, we get

$$\int_{(x)} C_{ij} x_j = \delta_{i0} \longrightarrow \int_{(x)} x_i = (C^{-1})_{i0} \tag{13}$$

It is not difficult to show that this solves all the eqs. (12) [5].

4 Examples

4.1 Paragrassmann Algebras

A Paragrassmann algebra $\mathcal{G}_1^p$ of order p is generated by the symbol θ, such that $\theta^{p+1} = 0$, and its elements are given by $x_i = \theta^i$, $i = 0, \cdots, p$. The product rule is simply $\theta^i \theta^j = \theta^{i+j}$. It is not difficult to show [4] that the C matrix exists and that $C_{ij} = \delta_{i+j,p}$. This matrix has the properties $C^T = C$ and $C^2 = 1$. Therefore we get

$$\int_{(\theta)} \theta^i = \delta_{ip} \tag{14}$$

In particular, for $p=1$ (Grassmann algebra), we reproduce the Berezin's integration rules.

4.2 Matrix Algebras

Consider the algebra $\mathcal{A}_N$ of the $N \times N$ matrices. A basis is given by the set of matrices $\{e^{(nm)}\}$ such that $e_{ij}^{(nm)} = \delta_i^n \delta_j^m$. The product rule is $e^{(nm)} e^{(pq)} = \delta_{mp} e^{nq}$. Also in this case the C matrix exists [5] and it is such that under its action $e^{(nm)} \to e^{(mn)}$, that is $C_{(mn)(rs)} = \delta_{ms}\delta_{nr}$. It has the properties $C^T = C$ and $C^2 = 1$. Since the identity is given by $I = \sum_{n=1}^N e^{(nn)}$ the integration rule gives

$$\int_{(e)} e^{(rs)} = \sum_n (C^{-1})_{(nn)(rs)} = \delta_{rs} \tag{15}$$

For an arbitrary matrix A, it follows

$$\int_{(e)} A = \sum_{nm=1}^{N} a_{nm} \int_{(e)} e^{(nm)} = Tr(A) \tag{16}$$

4.3 Projective Group Algebras

Given a group G and an arbitrary projective linear representation $\mathcal{A}(G)$, $a \to x(a)$, with $a \in G$ and $x(a) \in \mathcal{A}(G)$, the vector space $\mathcal{A}(G)$ has naturally an algebra structure given by the product $x(a)x(b) = \exp(i\alpha(a,b))x(ab)$, where the phase $\alpha(a,b)$ is the cocycle associated to the projective representation. Also this algebra is self-conjugated[6] with C mapping $x(a)$ into $x(a^{-1})$, that is $C_{ab} = \delta_{ab,e}$, where e the identity in the group. The integration rules are

$$\int_{(x)} x(a) = \delta_{e,a} \tag{17}$$

In the case $G = R^D$, and $\mathcal{A}(G)$ a vector representation, one has $x(\vec{a}) = \exp(i\vec{q}\cdot\vec{a})$ and the algebraic integration coincides with the integration over the variables $\vec{q}$[6].

5 Derivations

There is an important theorem[5] relating the algebraic integration and a particular type of derivations on the self-conjugated algebras which generalizes the ordinary integration by part formula. The theorem says: *If D is a derivation such that $\int_{(x)} Df(x) = 0$ for any function $f(x)$ on the algebra, then the integral is invariant under the related automorphism $\exp(\alpha D)$, and viceversa.* If the matrix C satisfies $C^T = C$, as in the examples above, it is also possible to prove that the inner derivations give rise to automorphisms leaving invariant the integration measure. The inner derivations for an associative algebra are given by $D_a x_i = [x_i, a]$. Therefore, for a matrix algebra our theorems implies $\int_{(e)} D_A B = \int_{(e)} [B, A] = 0$, which is nothing but $Tr([B, A]) = 0$.

6 Integration over a subalgebra

Given a self-conjugated algebra $\mathcal{A}$ and a self-conjugated subalgebra $\mathcal{B}$, we would like to recover the integration over $\mathcal{B}$ in terms of the integration over $\mathcal{A}$. Since we have the decomposition $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$, we look for an element $P \in \mathcal{A}$ such that $\int_{\mathcal{A}} \mathcal{C}P = 0$ and $\int_{\mathcal{A}} \mathcal{B}P = \int_{\mathcal{B}} \mathcal{B}$, that is

$$\int_{\mathcal{A}} \mathcal{A}P = \int_{\mathcal{A}} \mathcal{B}P = \int_{\mathcal{B}} \mathcal{B} \tag{18}$$

Since there are as many conditions as the elements of $\mathcal{A}$, P is uniquely determined. As an example we can consider a Grassmann algebra $\mathcal{G}$ with a single generator θ. This can be embedded into the algebra $\mathcal{A}_2$, the algebra of the

2×2 matrices, by the correspondence $\theta \to \sigma_+$, and $1 \to 1_2$. Then, one finds $P = \sigma_-$ [5], and as a consequence one can express the Berezin's integral in terms of traces over 2×2 matrices

$$\int_{\mathcal{A}_2} f(\sigma_+)\sigma_- = \int_{(\theta)} f(\theta) = Tr[f(\sigma_+)\sigma_-] \tag{19}$$

Reproducing the known rules

$$\int_{(\theta)} 1 = Tr[\sigma_-] = 0, \qquad \int_{(\theta)} \theta = Tr[\sigma_+\sigma_-] = 0 \tag{20}$$

7 Conclusions and Outlook

We have shown that it is possible to define an integration according to the principles outlined in the introduction for a very large class of algebras (associative self-conjugated algebras with identity). In particular, we have recovered many known cases and shown that it is possible to generalize the theorem of integration by part valid for the Lebesgue integral. We have also noticed that the integration over subalgebras is defined in a natural way. Furthermore, this approach can be extended to algebras which are not self-conjugated (the right and left multiplication representations are not equivalent), as the algebra of the bosonic and of the q-oscillators [4]. Also the case of the nonassociative octonionic algebra can be treated by these methods [4]. The next step in this approach would be the definition of the path-integral via tensoring of the algebra.

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